Problem set 4, Problem 2
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Solution

a) Assume that a given $w \in W$ can be written as

$$w = u_1 + v_1$$
$$w = u_2 + v_2$$

for some $u_1, u_2 \in U$ and $v_1, v_2 \in V$.

Substract both expressions to get

$$0 = u_1 - u_2 + v_1 - v_2$$

i.e. $u_2 - u_1 = v_1 - v_2$.

But $u_2 - u_1 \in U$, $v_1 - v_2 \in V$ since $U + V$ are subspaces

$$\Rightarrow \quad u_2 - u_1 = v_1 - v_2 \in U \cap V = \{0\}$$

In other words, $u_2 - u_1 = 0 = v_1 - v_2$

i.e. $u_2 = u_1$

$v_1 = v_2$. ∎

COMMENTS • Note how I did not pick a basis. Many people did that. It is ok if you write it up well. But it is long. :)

• To make your life easier, ask yourselves:

→ How dependent is this result on my ability to pick a basis?

→ Do I really need to?

b) Pick a basis $\{v_1, \ldots, v_r\}$ for U.

We can extend this to a basis $\{v_1, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ of V.

Claim The subspace $U' = \text{span}(v_{r+1}, \ldots, v_n)$ satisfies

$$W = U \oplus U'.$$

Pf/ Since $\{v_1, \ldots, v_n\}$ is l.i., then

$$U \cap U' = \text{span}(v_1, \ldots, v_r) \cap \text{span}(v_{r+1}, \ldots, v_n) = \{0\}$$

Furthermore, $W = U + U'$ since $\{v_1, \ldots, v_n\}$ is a basis.

Hence, $W = U \oplus U'$ Done! :)

* U' is actually isomorphic to V/U =)

COMMENTS: Many people started off by picking a basis for V and trying to pinpoint the basis elements for U. This is ok if you wrote it up well. Note that U' is NOT $V - U$.* ∎